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RELATIVE OPERATOR ENTROPY AND KARCHER MEAN (Research on structure of operators by order and geometry with related topics)

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RELATIVE OPERATOR ENTROPY AND KARCHER MEAN

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1. INTRODUCTION

Lawson and Lim [10] showed that the Karcher equation for positive invertible operators A_j ($j = 1, 2, \dots, n$), X on a Hilbert space and a weight $\{\omega_j\}$ ($\omega_j \geq 0, \sum_j \omega_j = 1$):

$$(KE) \quad 0 = \sum_{j=1}^n \omega_j \log \left(X^{-\frac{1}{2}} A_j X^{-\frac{1}{2}} \right)$$

has a unique positive invertible solution

$$X = G_K(\omega_j; A_j) = G_K(\omega; \mathbb{A})$$

for $\omega = (\omega_1, \dots, \omega_n)$ and $\mathbb{A} = (A_1, \dots, A_n)$. It is called the (weighted n -variable) Karcher mean. This definition depends on the invertibility of operators: Their theory depends on the Thompson metric $d(A, B) = \|\log(A^{-1/2}BA^{-1/2})\|$ and the power operator mean corresponding to the power function $f_{m,r,t}(x) = (1 - t + tx^r)^{\frac{1}{r}}$: The Karcher mean of positive invertible operators coincides with the strong-operator limit of the power means $P_t(\omega; \mathbb{A})$ as $t \rightarrow 0$:

$$G_K(\omega; \mathbb{A}) = \text{s-lim}_{t \rightarrow 0} P_t(\omega; \mathbb{A}).$$

Thus it needs substantially the invertibility of positive operators.

In this note, we extend it to a mean for (non-invertible) positive operators by virtue of the relative operator entropy. So, we observe the properties for the relative operator entropy with the existence conditions which are closely related the kernels and ranges for operators. Defining the Karcher mean for non-invertible operators as the strong-operator limit, we discuss their properties and kernels. Then we see that the original Kubo-Ando geometric mean satisfies the Karcher equation with the kernel condition. Also we verify that the Karcher mean for operators with the closed ranges is a unique solution of this equation.

2. RELATIVE OPERATOR ENTROPY

First we review the *relative operator entropy* $S(A|B)$ for positive (bounded linear) operators A, B on a Hilbert space, see [5, 6, 7, 2, 3, 4].

Nakamura and Umegaki [11] extended the notion of the entropy formulated by von Neumann and gave the entropy by $-A \log A$ for a positive operator A on a Hilbert space. Also, Umegaki [13] introduced the relative entropy as a noncommutative version of the Kullback-Leibler entropy, which is given by the trace of $A \log A - A \log B$ for positive operators A, B affiliated with a semifinite von Neumann algebra. In [5], Fujii and Kamei

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introduced the relative operator entropy which is a relative version of the operator entropy defined by Nakamura-Umegaki: Let A and B be positive operators on a Hilbert space H . If B is invertible, then the relative operator entropy is defined by

$$S(A|B) = B^{\frac{1}{2}} \eta \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \right) B^{\frac{1}{2}}$$

where η is the entropy function:

$$\eta(x) = -x \log x \quad \text{if } x > 0, \quad \eta(0) = 0.$$

In addition, if A is invertible, then the relative operator entropy is rewritten as follows:

$$S(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

In the case, since $\log t$ is operator monotone, $S(A|B)$ has the right monotonicity:

$$B \leq C \quad \text{implies} \quad S(A|B) \leq S(A|C).$$

Moreover, by using the fact that $\lim_{t \rightarrow 0} \frac{x^t - 1}{t} = \log x$, it is constructed by Uhlmann's way [12]:

$$S(A|B) = \text{s-lim}_{t \rightarrow 0} \frac{A \#_t B - A}{t},$$

where the geometric operator mean $A \#_t B$ is defined by

$$A \#_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2}.$$

Next, let A and B be non-invertible. For each $\varepsilon > 0$, since $B + \varepsilon$ is invertible, the relative operator entropy $S(A|B + \varepsilon)$ exists and

$$S(A|B + \varepsilon) = \text{s-lim}_{\delta \downarrow 0} S(A + \delta | B + \varepsilon)$$

in the strong operator topology. Since for each $\delta > 0$, $S(A + \delta | B + \varepsilon)$ has the right-term monotone decreasing property as $\varepsilon \downarrow 0$, we have

$$S(A + \delta | B + \varepsilon) \geq S(A + \delta | B + \varepsilon') \quad \text{for } \varepsilon > \varepsilon' > 0$$

and so $S(A|B + \varepsilon) \geq S(A|B + \varepsilon')$ as $\delta \downarrow 0$. Therefore, in the case of non-invertible A and B , we can define

$$S(A|B) = \text{s-lim}_{\varepsilon \downarrow 0} S(A|B + \varepsilon)$$

if the strong-operator limit exists as a bounded operator. In fact, by the right-term monotonicity, the lower boundedness for $\{S(A|B + \varepsilon) | \varepsilon > 0\}$ guarantees the strong-operator limit. But, in general, $S(A|B)$ does not always exist. For example, $S(A|B)$ does not exist for invertible A and non-invertible B .

Here, we state a existence condition of the relative operator entropy:

Lemma 2.1. *Let A, B be positive operators. The relative operator entropy $S(A|B)$ exists if and only if $E(\alpha) = (-1 - \log \alpha)A + \alpha B$ is bounded below for all $\alpha > 0$.*

Next, we consider another equivalent definition of Uhlmann's type. For this, based on the fact that $\frac{x^t - 1}{t} \searrow \log x$ as $t \downarrow 0$, it follows that $\frac{A \#_t B - A}{t}$ is monotone-decreasing as $t \downarrow 0$, so that another equivalent definition of Uhlmann's type is the derivative one for the path of geometric means $A \#_t B$ for $t \in (0, 1]$:

$$S(A|B) = \text{s-lim}_{t \downarrow 0} \frac{A \#_t B - A}{t}$$

if the strong-operator limit exists. In fact, put $X = s\text{-}\lim_{t \downarrow 0} \frac{A \#_t B - A}{t}$ and since

$$\frac{A \#_t B - A}{t} \leq \frac{(A + \varepsilon) \#_t (B + \varepsilon) - (A + \varepsilon)}{t} \quad \text{for each } \varepsilon > 0,$$

we have $X \leq s\text{-}\lim_{t \downarrow 0} \frac{(A + \varepsilon) \#_t (B + \varepsilon) - (A + \varepsilon)}{t} = S(A + \varepsilon | B + \varepsilon)$ as $t \downarrow 0$ and so $X \leq S(A | B)$ as $\varepsilon \downarrow 0$. On the other hand, since

$$\frac{(A + \varepsilon) \#_t (B + \varepsilon) - (A + \varepsilon)}{t} \geq S(A + \varepsilon | B + \varepsilon) \quad \text{for each } \varepsilon > 0 \text{ and } t \in (0, 1],$$

we have $\frac{A \#_t B - A}{t} \geq S(A | B)$ as $\varepsilon \downarrow 0$ for $t > 0$ and so $X \geq S(A | B)$. Therefore, $X = S(A | B)$ and we have another equivalent definition of Uhlmann's type for non-invertible case.

If A and B are commuting and $S(A | B)$ is defined, then

$$S(A | B) = A \log B - A \log A,$$

in particular, $S(0 | B) = 0$ for all $B \geq 0$. In fact, since $S(A | B)$ exists, by Lemma 2.1, there exists a scalar $c \in \mathbb{R}$ such that for all $\alpha > 0$

$$\begin{aligned} c &\leq (-1 - \log \alpha)A + \alpha B \\ &\leq (-1 - \log \alpha)A + \alpha(B + \varepsilon) \quad \text{for each } \varepsilon > 0 \end{aligned}$$

and so

$$\begin{aligned} c &\leq (B + \varepsilon)(-(B + \varepsilon)^{-1}A \log(B + \varepsilon)^{-1}A) \\ &= -A \log A + A \log(B + \varepsilon). \end{aligned}$$

Therefore, since $A \log B - A \log A$ exists, we have $A \log B - A \log A = S(A | B)$.

Under the existence, we have the following properties of $S(A | B)$ for positive operators A and B by those for operator means:

Lemma 2.2. *Under the existence, the following properties like operator means hold:*

- (1) **right monotonicity:** If $B \leq B'$, then $S(A | B) \leq S(A | B')$.
- (2) **transformer inequality:** $T^*S(A | B)T \leq S(T^*AT | T^*BT)$ for all T
(the equality holds for invertible T).
- (2') **informational monotonicity:** $\Phi(S(A | B)) \leq S(\Phi(A) | \Phi(B))$
for all normal positive linear maps Φ .
- (3) **sub-additivity:** $S(A_1 | B_1) + S(A_2 | B_2) \leq S(A_1 + A_2 | B_1 + B_2)$.
- (3') **joint concavity:** $(1-t)S(A_1 | B_1) + tS(A_2 | B_2) \leq S((1-t)A_1 + tA_2 | (1-t)B_1 + tB_2)$ for all $t \in [0, 1]$.
- (4) **upper bound:** $S(A | B) \leq B - A$.
- (5) **kernel inclusion:** $\ker S(A | B) \supset \ker A$.
- (6) **orthogonality:** $S(\bigoplus_k A_k | \bigoplus_k B_k) = \bigoplus_k S(A_k | B_k)$.
- (7) **affine parametrization:** $S(A | A \#_t B) = t S(A | B)$ for all $t \in [0, 1]$.

Here we recall the equality condition for the transformer inequality (2) of Lemma 2.2:

Lemma 2.3. *Let A and B be positive operators. Under the existence of $S(A | B)$, the transformer equality*

$$T^*S(A | B)T = S(T^*AT | T^*BT)$$

holds for an operator T with $\ker T^ \subset \ker A \cap \ker B$.*

We have the following relations around the existence condition for the relative operator entropy:

Theorem 2.4. *The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold in the following conditions for a pair of $A, B \geq 0$ and each converse does not always hold.*

- (1) **majorization or range inclusion:** $\exists \alpha > 0; A \leq \alpha B$, i.e., $\text{ran } A^{\frac{1}{2}} \subset \text{ran } B^{\frac{1}{2}}$.
- (2) **existence condition:** $S(A|B)$ exists as a bounded operator, i.e.,

$$\inf_{\alpha > 0} \left[\frac{1}{\alpha} B - A + (\log \alpha) A \right] > -\infty.$$

- (3) **B -absolute continuity:** $A = [B]A \left(= A^{\frac{1}{2}} P_M A^{\frac{1}{2}} = \lim_{t \downarrow 0} A \#_t B \right)$.
- (4) **kernel inclusion:** $\ker A \supset \ker B$.

Remark 2.1. If both ranges of A and B are closed, in particular, for the case of matrices, the above conditions in Theorem 2.4 are all equivalent since the relation $\text{ran } A^{\frac{1}{2}} = \overline{\text{ran } A} = (\ker A)^{\perp}$ holds for all positive operators A .

For invertible positive operators A and B , it is easy to see that the positivity (resp. negativity) of $S(A|B)$ is equivalent to $B \geq A$ (resp. $A \geq B$) and hence $S(A|B) = 0$ if and only if $A = B$. Second we discuss non-invertible case:

Theorem 2.5. *Let A and B be positive operators. Suppose $S(A|B)$ exists. $S(A|B) \geq 0$ if and only if $B \geq A$. If $\ker A$ is trivial, then $S(A|B) \leq 0$ if and only if $A \geq B$. Consequently, for A with the trivial kernel, $S(A|B) = 0$ if and only if $A = B$.*

3. KARCHER MEAN FOR NON-INVERTIBLE POSITIVE OPERATORS

For non-invertible positive operators A_j ($j = 1, 2, \dots, n$), for each $\varepsilon > 0$ the Karcher mean $X_\varepsilon = G_K(\omega_j; A_j + \varepsilon) \geq 0$ exists and the monotonicity of G_K guarantees the strong-operator limit:

$$X_0 = \text{s-lim}_{\varepsilon \rightarrow 0} X_\varepsilon = \text{s-lim}_{\varepsilon \rightarrow 0} G_K(\omega_j; A_j + \varepsilon).$$

Naturally we write $X_0 = G_K(\omega_j; A_j)$ for non-invertible A_j and call it *the Karcher mean* again.

Here we extend the extremal means with a weight $\{\omega_j\}$ synchronously to G_K : The arithmetic mean A and the harmonic one H for non-invertible A_j are defined by

$$A(\omega_j; A_j) = \sum_j \omega_j A_j$$

$$H(\omega_j; A_j) = \text{s-lim}_{\varepsilon \rightarrow 0} H(\omega_j; A_j + \varepsilon) = \text{s-lim}_{\varepsilon \rightarrow 0} \left(\sum_j \omega_j (A_j + \varepsilon)^{-1} \right)^{-1}.$$

As for this construction of corresponding mean, we say ' H is the adjoint of A ' as in the Kubo-Ando theory.

Theorem 3.1. *For a weight $\{\omega_j\}$ and positive operators A_j ($j = 1, 2, \dots, n$), the following properties hold:*

- (1) **monotonicity:** If $A_j \leq B_j$ for all j , then $G_K(\omega_j; A_j) \leq G_K(\omega_j; B_j)$.
- (2) **transformer inequality:** $T^* G_K(\omega_j; A_j) T \leq G_K(\omega_j; T^* A_j T)$ for all T
(the equality holds for invertible T).

- (2') **informational monotonicity:** $\Phi(G_K(\omega_j; A_j)) \leq G_K(\omega_j; \Phi(A_j))$
for all normal positive linear maps Φ .
- (3) **sub-additivity:** $G_K(\omega_j; A_j) + G_K(\omega_j; B_j) \leq G_K(\omega_j; A_j + B_j)$.
- (3') **joint concavity:**
 $(1-t)G_K(\omega_j; A_j) + tG_K(\omega_j; B_j) \leq G_K(\omega_j; (1-t)A_j + tB_j)$ for all $t \in [0, 1]$.
- (4) **consistency with scalars:**
If all A_j are commuting, then $G_K(\omega_j; A_j) = \prod_{j=1}^n A_j^{\omega_j}$ with convention $A^0 = I$.
- (5) **self-adjointness:** $G_K(\omega_j; A_j) = s\text{-}\lim_{\varepsilon \downarrow 0} G_K(\omega_j; (A_j + \varepsilon)^{-1})^{-1}$.
- (6) **joint homogeneity:** $G_K(\omega_j; c_j A_j) = \prod_{j=1}^n c_j^{\omega_j} G_K(\omega_j; A_j)$
for non-negative numbers $c_j \geq 0$ ($j = 1, 2, \dots, n$).
- (7) **AGH weighted mean inequality:**

$$H(\omega_j; A_j) \leq G_K(\omega_j; A_j) \leq A(\omega_j; A_j).$$

(8) **orthogonality:** $G_K\left(\omega_j; \bigoplus_m A_{j,m}\right) = \bigoplus_m G_K(\omega_j; A_{j,m})$.

Note that if $\omega_k = 0$ for some k , then n -mean $G_K(\omega_j; A_j)$ is nothing but $(n-1)$ -mean without ω_k, A_k . So we call $G_K(\omega_j; A_j)$ the *proper* Karcher mean if $\omega_j > 0$ for all j . Then we also call the weight $\{\omega_j\}$ *proper*.

Lemma 3.2. For a proper weight $\{\omega_j\}$ and positive operators A_j ($j = 1, 2, \dots, n$),

$$\text{ran } A(\omega_j; A_j)^{\frac{1}{2}} = \bigvee_j \text{ran } A_j^{\frac{1}{2}} \quad \text{and} \quad \text{ran } H(\omega_j; A_j)^{\frac{1}{2}} = \bigcap_j \text{ran } A_j^{\frac{1}{2}}.$$

By Lemma 3.2 and the transformer inequality (2) of Theorem 3.1, we have the following kernel condition for the Karcher mean.

Theorem 3.3. Let A_j be positive operators for $j = 1, 2, \dots, n$. For a proper weight $\{\omega_j\}$,

$$\ker G_K(\omega_j; A_j) = \bigvee_j \ker A_j.$$

For any projections P and Q , it is known in [9] that $P \#_t B = P \wedge Q$ for $t \in [0, 1]$. The following result is an extension of 2-variable case:

Corollary 3.4. For a proper weight $\{\omega_j\}$,

$$G_K(\omega_j; P_j) = \bigwedge_j P_j \quad \text{for projections } P_j \text{ } (j = 1, 2, \dots, n).$$

In this invertible case, note that (KE) is equivalent to a simple equation by the relative operator entropy

$$(**) \quad 0 = \sum_{j=1}^n \omega_j S(X|A_j),$$

which also makes sense for non-invertible A_j . But this equation always has a trivial solution $X = 0$ since $S(0|A_j) = 0$. For the case of Corollary 3.4, the entropy is $S(P|P_j) =$

$P \log P_j - P \log P = 0$, and hence P is indeed a solution of the equation (**). But this consideration shows that each projection Q with $0 \leq Q \leq P$ is a solution of (**).

Thereby, a reasonable extension of (KE) is the following EKE (*Extended Karcher equation*) with the kernel condition under the existence of each $S(X|A_j)$:

$$(EKE) \quad 0 = \sum_{j=1}^n \omega_j S(X|A_j) \quad \text{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

Remark 3.1. If operators A_j are commuting for all j , then $X_0 = G_K(\omega_j; A_j) = \prod_j A_j^{\omega_j}$ and X_0 is a solution of (EKE). But, if the kernel condition is removed, the following example gives another solution X even if X commutes with all A_j .

Example 1. For diagonal matrices $A = \text{diag}(a, b, c, 0)$ and $B = \text{diag}(\frac{1}{a}, b, 0, d)$, take $X_1 = \text{diag}(0, b, 0, 0)$. Then, all matrices are commuting and hence

$$\begin{aligned} S(X_1|A) + S(X_1|B) &= -2X_1 \log X_1 + X_1 \log A + X_1 \log B \\ &= \text{diag}(0, -2b \log b, 0, 0) + \text{diag}(0, b \log b, 0, 0) + \text{diag}(0, b \log b, 0, 0) = 0. \end{aligned}$$

So X_1 is a solution while $X_1 \neq X_0 = \text{diag}(1, b, 0, 0)$.

Remark 3.2. For the case of projections $A_j = P_j$, the above P is a unique solution of (EKE). In fact, suppose Y is another solution. Then the kernel condition $P \leq P_j$ shows $PYP = Y$ and hence we have $YP_j = P_jY = Y$ and $Y \log P_j = YP_j \log P_j = 0$. Therefore

$$\begin{aligned} 0 &= \sum_j \omega_j S(Y|P_j) = \sum_j \omega_j (Y \log P_j - Y \log Y) \\ &= \sum_j \omega_j (-Y \log Y) = -Y \log Y, \end{aligned}$$

which implies that Y must be a projection and consequently $Y = P$ by the kernel condition.

Moreover note that $S(A|B)$ does not always exist as a bounded self-adjoint operator as in the preceding section. But $S(X_0|A_j)$ indeed exists for the Karcher mean $X_0 = G_K(\omega_j; A_j)$:

Lemma 3.5. Let A_j be positive operators and $\{\omega_j\}$ a weight. For the Karcher mean $X_0 = G_K(\omega_j; A_j)$, each entropy $S(X_0|A_j)$ exists for $\omega_j > 0$. For $\omega_k > 0$, bounds are expressed by

$$-\frac{M_k}{\omega_k} \leq S(X_0|A_k) \leq M_k$$

for $M_k = \max_{j \neq k} \|A_j\| + 1$.

Thus far, we have not showed that our Karcher mean $X_0 = \text{s-lim}_{\epsilon \rightarrow 0} X_\epsilon$ satisfies (EKE) for general positive operators. Here we can obtain only the inequality:

Lemma 3.6. Let A_j be positive operators for $j = 1, 2, \dots, n$ and $\{\omega_j\}$ a weight. Then

$$0 \leq \sum_{j=1}^n \omega_j S(X_0|A_j) \quad \text{with} \quad \ker X_0 = \bigvee_{\omega_j > 0} \ker A_j.$$

However, in the case of $n = 2$, we have the following result:

Theorem 3.7. For positive operators A and B and $t \in (0, 1)$, the original geometric mean $A \#_t B$ satisfies (EKE).

Recall that a non-invertible positive operators A has the closed range if and only if 0 is an isolated point in $\sigma(A)$. Any positive semi-definite matrix has the closed range. Finally in this section, we show that the Karcher mean for such operators is a unique solution of (EKE). To see this, we verify the following fact:

Lemma 3.8. If A_j ($j = 1, 2, \dots, n$) are positive operators whose ranges are closed, then so is $X_0 = G_K(\omega_j; A_j)$.

So we have a unique solution of (EKE):

Theorem 3.9. If A_j are positive operators whose ranges are closed for $j = 1, 2, \dots, n$, the Karcher mean $X_0 = G_K(\omega_j; A_j)$ is a unique solution of (EKE).

Therefore, we have the following result in the matrix case:

Corollary 3.10. For positive semi-definite matrices A_j for $j = 1, 2, \dots, n$ and a weight $\{\omega_j\}$, the Karcher mean $X = G_K(\omega_j; A_j)$ is the unique solution of (EKE):

$$0 = \sum_{j=1}^n \omega_j S(X|A_j) \quad \text{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

4. POWER MEANS FOR NON-INVERTIBLE CASE

Let A_j be positive operators for $j = 1, 2, \dots, n$ and $\{\omega_j\}$ a weight. For each $\varepsilon > 0$, similarly to the Karcher mean $X_\varepsilon = G_K(\omega_j; A_j + \varepsilon)$, the power means $P_t(\omega_j; A + \varepsilon)$ for $t \in (0, 1]$ is the unique positive invertible solution of the power mean equation

$$X = \sum_{j=1}^n \omega_j (X \#_t (A_j + \varepsilon)).$$

Then the Karcher mean for invertible case is the strong-operator limit of the power means:

$$\text{s-lim}_{\varepsilon \downarrow 0} P_t(\omega_j; A_j + \varepsilon) = X_\varepsilon.$$

The power means $P_t(\omega_j; A_j + \varepsilon)$ are monotone decreasing for $\varepsilon \downarrow 0$ by [10, Proposition 3.6 (4)] and lower bounded by the zero operator. Hence $P_t(\omega_j; A_j) = \inf_{\varepsilon > 0} P_t(\omega_j; A_j + \varepsilon)$ exists and

$$P_t(\omega_j; A_j) = \text{s-lim}_{\varepsilon \downarrow 0} P_t(\omega_j; A_j + \varepsilon)$$

in the strong operator topology and so $P_t(\omega_j; A_j)$ is a solution of the power mean equation

$$(1) \quad X = \sum_j \omega_j (X \#_t A_j)$$

for $t \in (0, 1]$.

Similarly to the Karcher equation for positive operators, the power mean equation (1) always has a trivial solution $X = 0$. Thereby, we consider the following EPE (*Extended Power mean equation*) with the kernel condition:

$$(EPE) \quad X = \sum_j \omega_j X \#_t A_j, \quad \text{with} \quad \ker X = \bigcap_{\omega_j > 0} \ker A_j.$$

Theorem 4.1. Let A_j be positive operators and $\{\omega_j\}$ a weight. Then the power means $P_t(\omega_j; A_j)$ for $t \in (0, 1]$ satisfy (EPE) and

$$P_t(\omega_j; A_j) \searrow G_K(\omega_j; A_j) \quad \text{as } t \downarrow 0.$$

Theorem Y (Yamazaki [14]). Let A_j and X be positive operators for $j = 1, 2, \dots, n$ and $\{\omega_j\}$ a weight. Then

$$\sum_j \omega_j S(X|A_j) \geq 0 \quad \text{implies} \quad G_K(\omega_j; A_j) \geq X.$$

Moreover, if A_j and X are invertible, then

$$\sum_j \omega_j S(X|A_j) \leq 0 \quad \text{implies} \quad G_K(\omega_j; A_j) \leq X.$$

The following result follows from the uniqueness of the power means and is an extension of Theorem Y:

Theorem 4.2. Let A_j and X be positive operators for $j = 1, 2, \dots, n$ and a weight $\{\omega_j\}$. Then

$$X \leq \sum_j \omega_j X \#_t A_j \quad \text{implies} \quad X \leq P_t(\omega_j; A_j).$$

Moreover, if A_j and X are invertible, then

$$X \geq \sum_j \omega_j X \#_t A_j \quad \text{implies} \quad X \geq P_t(\omega_j; A_j).$$

Remark 4.1. Theorem 4.2 is an extension of Theorem Y. Indeed, suppose $\sum_j \omega_j S(X|A_j) \geq 0$. Then

$$0 \leq \sum_j \omega_j S(X|A_j) \leq \frac{\sum_j \omega_j X \#_t A_j - X}{t}$$

and it follows from Theorem 4.2 that $X \leq P_t(\omega_j; A_j)$ for all $0 < t \leq 1$. Taking limit as $t \downarrow 0$, we have $X \leq G_K(\omega_j; A_j)$. Another part is obtained by this result.

Remark 4.2. Finally, considering the power means for negative parameters as the adjoint in the sense of Kubo-Ando theory, for invertible case, we can show that the uniqueness of the Karcher equation follows from the case of power means without the implicit function theorem. See [8] for more detail.

Though the properties for power means are supplied, the following general problems are not still answered:

Conjecture. For non-invertible positive operators on a Hilbert space, the Karcher mean satisfies (EKE) and it is a unique solution of (EKE).

Conjecture 2. For non-invertible positive operators on a Hilbert space, each power mean $P_t(\omega_j; A_j)$ for $t \in (0, 1]$ is a unique solution of (EPE).

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